ON PRESENTATIONS OF INTEGER POLYNOMIAL POINTS OF SIMPLE GROUPS OVER NUMBER FIELDS

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In this paper we prove the following

Theorem 1. *Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $\mathbf{G}$ be a connected, noncommutative, absolutely almost simple algebraic K-group. If the K-rank of $\mathbf{G}$ equals 2, then $\mathbf{G}(\mathcal{O}_K[t])$ is not finitely presented.*

Actually, we will prove a slightly stronger version of Theorem 1 by showing that if $\mathbf{G}(\mathcal{O}_K[t])$ is as in Theorem 1, then $\mathbf{G}(\mathcal{O}_K[t])$ is not of type FP_2.

0.1. **Related results.** Krstić-McCool proved that $\mathrm{GL}_3(A)$ is not finitely presented if there is an epimorphism from A to $F[t]$ for some field F [K-M].

Suslin proved that $\mathrm{SL_n}(A[t_1, \ldots, t_k])$ is generated by elemetary matrices if $n \geq 3$, A is a regular ring, and $K_1(A) \cong A^\times$ [Su]. Grunewald-Mennicke-Vaserstein proved that $\mathrm{Sp_{2n}}(A[t_1, \ldots, t_k])$ is generated by elementary matrices if $n \geq 2$ and A is a Euclidean ring or a local principal ideal ring [G-M-V].

In Bux-Mohammadi-Wortman, it's shown that $\mathrm{SL_n}(\mathbb{Z}[t])$ is not of type FP_{n-1} [B-M-W]. The case when $n = 3$ is a special case of Theorem 1.

While most of the results listed above allow for more general rings than $\mathcal{O}_K[t]$, the result of this paper, and the techniques used to prove it, are distinguished by their applicability to a class of semisimple groups that extends beyond special linear and symplectic groups.

1. Preliminary and notation

Throughout the remainder, we let $\mathbf{G}$ be as in Theorem 1 and we let $\Gamma = \mathbf{G}(\mathcal{O}_K[t])$.

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Let L be an algebraically closed field containing $K((t^{-1}))$ fixed once and for all. In the the sequel the Zarsiki topology is defined with this fixed algebraically closed field in mind.

Let $\mathbf{S}$ be a maximal K-split torus of $\mathbf{G}$. Let $\{\alpha, \beta\}$ be a set of simple K-roots for $(\mathbf{G}, \mathbf{S})$, and define $\mathbf{T} = (\ker(\alpha))^\circ$, the connected component containing the identity.

Let $\mathbf{P}$ be a maximal K-parabolic subgroup of $\mathbf{G}$ that has $Z_{\mathbf{G}}(\mathbf{T})$ as a Levi subgroup where $Z_{\mathbf{G}}(\mathbf{T})$ denotes the centralizer of $\mathbf{T}$ in $\mathbf{G}$. Let $\mathbf{U}$ be unipotent radical of $\mathbf{P}$. We have $\mathbf{P} = \mathbf{U}Z_{\mathbf{G}}(\mathbf{T})$. We can further write

$$\mathbf{P} = \mathbf{UHMT}$$

where $\mathbf{H} \leq Z_{\mathbf{G}}(\mathbf{T})$ is a simple K-group of K-rank 1 and $\mathbf{M}$ is a K-anisotropic torus contained in the center of $Z_{\mathbf{G}}(\mathbf{T})$.

If $x \in K((t^{-1}))$ is algebraic over K then $x \in K$, hence $\mathbf{G}$ has $K((t^{-1}))$-rank 2 as well and $\mathbf{P}$ is a $K((t^{-1}))$-maximal parabolic of $\mathbf{G}$. It also follows that $\mathbf{H}$ has $K((t^{-1}))$-rank 1 and that $\mathbf{M}$ is $K((t^{-1}))$-anisotropic.

We let G, S, P, U, M, H and T denote the $K((t^{-1}))$-points of $\mathbf{G}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{U}$, $\mathbf{M}$, $\mathbf{H}$, and $\mathbf{T}$, respectively.

Let X denote the Bruhat-Tits building associated to G. This is a 2-dimensional simplicial complex, and the apartments (maximal flats) correspond to maximal $K((t^{-1}))$-split tori.

We fix once and for all a K-embedding of $\mathbf{G}$ in some $\mathbf{SL}_n$. Using this embedding we realize $\mathbf{G}(K[t])$ and Γ as subgroups of $\mathbf{SL}(K[t])$ and $\mathbf{SL}(\mathcal{O}_K[t])$ respectively. This embedding also gives an isometric embedding of X into $\tilde{A}_{n-1}$, the building of $\mathbf{SL}_n(K((t^{-1})))$; see [La].

2. Stabilizers of the Γ-action on its Euclidean building

Lemma 2. *If X is the Euclidean building for G, then the Γ stabilizers of cells in X are FP_m for all m.*

Proof. We first recall the proof of [B-M-W, Lemma 2]. Let $x_0 \in \tilde{A}_{n-1}$ be the vertex stabilized by $\mathbf{SL}_n(K[[t^{-1}]])$. We denote a diagonal matrix in $\mathbf{GL_n}(K((t^{-1})))$ with entries $s_1, s_2, ..., s_n \in K((t^{-1}))^\times$ by $D(s_1, s_2, ..., s_n)$, and we let $\mathfrak{S} \subseteq \tilde{A}_{n-1}$ be the sector based at x_0 and containing vertices of the form $D(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq ... \geq m_n$.

The sector $\mathfrak{S}$ is a fundamental domain for the action of $\mathbf{SL_n}(K[t])$ on $\tilde{A}_{n-1}$ (see [So]). In particular, for any vertex $z \in \tilde{A}_{n-1}$, there is some $h'_z \in \mathbf{SL_n}(K[t])$ and some integers $m_1 \geq m_2 \geq ... \geq m_n$ with $z = h'_z D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$. We let $h_z = h'_z D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})$.

For any $N \in \mathbb{N}$, let W_N be the $(N+1)$-dimensional vector space

$$W_N = \{\, p(t) \in \mathbb{C}[t] \mid \deg\big(p(t)\big) \le N\}$$

which is endowed with the obvious $K-$structure. If $N_1, \cdots, N_{n^2}$ in $\mathbb{N}$ are arbitrary then let

$$\mathbf{G}_{\{N_1,\cdots,N_{n^2}\}} = \{\mathbf{x} \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(\mathbf{x}) = 1\}$$

where $\det(\mathbf{x})$ is a polynomial in the coordinates of $\mathbf{x}$. To be more precise this is obtained from the usual determinant function when one considers the usual $n \times n$ matrix presentation of $\mathbf{x}$, and calculates the determinant in $\mathbf{Mat}_n(\mathbb{C}[t])$.

For our choice of vertex $z \in \tilde{A}_{n-1}$ above, the stabilizer of z in $\mathbf{SL}_n(K((t^{-1})))$ equals $h_z \mathbf{SL}_n(K[[t^{-1}]]) h_z^{-1}$. And with our fixed choice of h_z, there clearly exist some $N_i^z \in \mathbb{N}$ such that the stabilizer of the vertex z in $\mathbf{SL}_n(K[t])$ is $\mathbf{G}_{\{N_1^z,\cdots,N_{n^2}^z\}}(K)$. Furthermore, conditions on N_i^z force a group structure on $\mathbf{G}_z = \mathbf{G}_{\{N_1^z,\cdots,N_{n^2}^z\}}$. Therefore, the stabilizer of z in $\mathbf{SL}_n(K[t])$ is the K-points of the affine K-group $\mathbf{G}_z$, and the stabilizer of z in $\mathbf{SL}_n(\mathcal{O}_K[t])$ is $\mathbf{G}_z(\mathcal{O}_K)$.

Let σ be a cell in $\tilde{A}_{n-1}$. The action of $\mathbf{SL}_n(K[t])$ on $\tilde{A}_{n-1}$ is type preserving, so if $\sigma \subset \mathfrak{S}$ is a simplex with vertices $z_1, z_2, ..., z_m$, then the stabilizer of σ in $\mathbf{SL}_n(\mathcal{O}_K[t])$ is

$$\big(\mathbf{G}_{z_1} \cap \cdots \cap \mathbf{G}_{z_m}\big)(\mathcal{O}_K)$$

Which implies that the stabilizer of σ in Γ is $\mathbf{G}_\sigma(\mathcal{O}_K)$ where $\mathbf{G}_\sigma = \mathbf{G} \cap \mathbf{G}_{z_1} \cap \cdots \cap \mathbf{G}_{z_m}$.

If $\psi \subset X$ is a cell, then we let $\sigma_1, \ldots, \sigma_k$ be simplices of $\tilde{A}_{n-1}$ such that their union contains ψ, and such that their union is contained in the union of any other set of simplices of $\tilde{A}_{n-1}$ that contains ψ.

The group Γ may not act on X type-preservingly, but the stabilizer of ψ in Γ will contain a finite index subgroup that fixes ψ pointwise. Because Γ does act type-preservingly on $\tilde{A}_{n-1}$, we have that the stabilizer of ψ in Γ contains

$$\big(\mathbf{G}_{\sigma_1} \cap \cdots \cap \mathbf{G}_{\sigma_k}\big)(\mathcal{O}_K)$$

as a finite index subgroup. This is an arithmetic group, and Borel-Serre [B-S] proved that any such group is FP_m for all m.

□

3. An unbounded ray in $\Gamma\backslash X$

The group Γ does not act cocompactly on X. Our next lemma is a generalization of Mahler's compactness criterion, and it will help us

identify a ray in X whose projection to $\Gamma\backslash X$ is proper. Our proof is similar to [B-M-W, Lemma 11].

Lemma 3. *If $e \in X$, $a \in G$, $u \in \Gamma$ is nontrivial, and $a^{-n}ua^n \to 1$ as $n \to \infty$, then $\{\Gamma a^n e : n \geq 0\} \subset \Gamma\backslash X$ is unbounded.*

Proof. Since G acts on X with bounded point stabilizers, it suffices to show that $\{\Gamma a^n : n \geq 0\} \subset \Gamma\backslash G$ is unbounded.

If $\{\Gamma a^n : n \geq 0\}$ is bounded, then it is contained in a set ΓB where $B \subset G$ is a bounded set. Thus, for any a^n, we have $a^n = \gamma b$ for some $\gamma \in \Gamma$ and $b \in B$. Hence $a^{-n}ua^n = b^{-1}\gamma^{-1}u\gamma b$.

Because u is nontrivial, $\gamma^{-1}u\gamma \in \Gamma - 1$ is bounded away from 1, and thus $b^{-1}\gamma^{-1}u\gamma b$ is bounded away from 1. That's a contradiction. $\square$

4. An unbounded semisimple element in $\mathbf{H}(\mathcal{O}_K[t])$

Recall that $\mathbf{H}$ has $K((t^{-1}))$-rank 1 (and K-rank 1), hence the Bruhat-Tits building of H, which will be denoted by X_H, is a tree. Let $\mathbf{S}'$ be a maximal K-split, thus $K((t^{-1}))$-split, torus of $\mathbf{H}$ and let $\mathbf{Q}^+$ and $\mathbf{Q}^-$ be opposite K-parabolic subgroups of $\mathbf{H}$ with Levi subgroup $Z_{\mathbf{H}}(\mathbf{S}')$.

We denote the unpotent radical of $\mathbf{Q}^\pm$ as $R_u(\mathbf{Q}^\pm)$, and we let $Q^\pm = \mathbf{Q}^\pm(K((t^{-1})))$, $R_u(Q^\pm) = R_u(\mathbf{Q}^\pm)(K((t^{-1})))$, and $S' = \mathbf{S}'(K((t^{-1})))$.

See [Se, Proposition 25] for the next lemma.

Lemma 4. *Let $u^+ \in R_u(Q^+)$ and $u^- \in R_u(Q^-)$ and let $F^\pm = \mathrm{Fix}_{X_H}(u^\pm)$. Assume that $F^+ \cap F^- = \emptyset$. Then u^+u^- is a hyperbolic isometry of X_H.*

Proof. Let x be the midpoint between F^+ and F^-. Let p_1 be the path between x and F^+ and let p_2 be the path between x and F^-, and let ψ be an edge containing x, contained in $p_1 \cup p_2$, not contained in p_2, and oriented towards F^+.

Notice that $u^-p_2 \cup p_2$ is an embedded path between x and u^-x and that $p_1 \cup u^+p_1 \cup u^+p_2 \cup u^+u^-p_2$ is an embedded path between x and u^+u^-x. The edge $u^+u^-\psi$ is a continuation of the latter path that is oriented away from from both u^+u^-x and x.

If u^+u^- is elliptic, then it fixes the midpoint of the path between x and u^+u^-x and maps ψ to an oriented edge pointed towards x. Therefore, u^+u^- is hyperbolic.

$\square$

Lemma 5. *There exists elements $u^\pm \in R_u(\mathbf{Q}^\pm)(\mathcal{O}_K[t])$ of arbitrarily large norm.*

Proof. After perhaps replacing α with 2α, there is a root group $\mathbf{U}_\alpha \leq R_u(\mathbf{Q}^\pm)$ and a K-isomorphism of algebraic groups $f : \mathbb{A}^k \to \mathbf{U}_\alpha$ for some affine space $\mathbb{A}^k$.

The regular function f is defined by polynomials $f_i \in K[x_1, \ldots, x_k]$. Because f maps the identity element to the identity element, each f_i has a constant term of 0.

The field of fractions of $\mathcal{O}_K$ is K. We let N be the product of the denominators of the coefficients of the f_i. Then the image under f of the points $(Nt^j, \ldots, Nt^j)$ forms an unbounded sequence in j of points in $\mathbf{U}_\alpha(\mathcal{O}_K[t])$.

□

Lemma 6. *There exists a hyberbolic isometry $b \in \mathbf{H}(\mathcal{O}_K[t])$ of the tree X_H.*

Proof. Let $\ell' \subseteq X_H$ be the geodesic corresponding to S', and choose $u^\pm \in R_u(\mathbf{Q}^\pm)(\mathcal{O}_K[t])$ of sufficient norm such that $\ell' \cap F^+$ is disjoint from $\ell' \cap F^-$. Since F^+ and F^- are convex, and $\ell' - (F^+ \cup F^-)$ is the geodesic between them, it follows that $F^+ \cap F^- = \emptyset$. Now apply Lemma 4. □

5. Construction of cycles in X near Γ

Let $b \in \mathbf{H}(\mathcal{O}_K[t])$ be as in Lemma 6, and let $\mathbf{S}''$ be the $K((t^{-1}))$-split one dimensional torus corresponding to the axis of b in X_H. Define the $K((t^{-1}))$-split torus $\mathbf{A} = \langle \mathbf{S}'', \mathbf{T} \rangle \leq \mathbf{P}$ and let $A = \mathbf{A}(K((t^{-1})))$. Let $\mathcal{A}$ denote the apartment in X corresponding to A.

Recall that any unbounded element $a \in T$ translates $\mathcal{A}$, and that the axis for the translation is any geodesic in $\mathcal{A}$ that joins P with its opposite parabolic P^{op}, as usual $P^{op} = \mathbf{P}^{op}(K((t^{-1})))$ where $\mathbf{P}^{op}$ is the oppositie parabolic containing $Z_{\mathbf{G}}(\mathbf{T})$.

Note that b acts by translation on $\mathcal{A}$. In fact, b translates orthogonal to any geodesic in $\mathcal{A}$ that joins P with P^{op}. Indeed, choose an element w of the Weyl group with respect to $\mathbf{A}$ that reflects through a geodesic joining $\mathbf{P}$ and $\mathbf{P}^{op}$. Thus w fixes both parabolic groups, and their common Levi subgroup, and hence $\mathbf{H}$. Since $\mathbf{S}' = \mathbf{A} \cap \mathbf{H}$, w fixes $\mathbf{S}'$ and thus fixes any axis for b in $\mathcal{A}$. Therefore, either b translates orthogonal to any geodesic in $\mathcal{A}$ that joins P with P^{op}, or else b translates along a geodesic in $\mathcal{A}$ that joins P with P^{op}. The latter option would contradict Lemma 3 since for any $e \in \mathcal{A}$, we have $\Gamma b^n e = \Gamma e \in \Gamma \backslash X$ and yet there is an unbounded $a \in T$ such that the ray determined by $a^n e$ is parallel to the ray determined by $b^n e$ and yet $a^{-n} u a^n \to 1$ either for any $u \in \mathbf{U}(\mathcal{O}_K[t])$ or for any u in the $\mathcal{O}_K[t]$-points of the unipotent radical of $\mathbf{P}^{op}$.

The spherical Tits building for G and X is a graph, and the apartment $\mathcal{A}$ corresponds to a circle in the spherical Tits building. Suppose

this circle has vertices $P_1, \ldots, P_n$ and edges $Q_1, ..., Q_n$ where each $\mathbf{P}_i$ is a maximal proper $K((t^{-1}))$-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{A}$, each $\mathbf{Q}_i$ is a minimal $K((t^{-1}))$-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{A}$, and $\mathbf{P}_1 = \mathbf{P}$. We further assume that mod n, the edge Q_i has vertices P_i and P_{i+1}.

Notice that $\mathbf{U} \leq \mathbf{Q}_1 \cap \mathbf{Q}_n$ since $\mathbf{P} = \mathbf{P}_1$ contains both $\mathbf{Q}_1$ and $\mathbf{Q}_n$. That is, any element of $\mathbf{U}(\mathcal{O}_K)$ fixes the edges Q_1 and Q_n.

Let $\mathbf{U}_1$ be the root group corresponding to the half circle that contains Q_1 but not Q_2, so that $\mathbf{U}_1 \leq \mathbf{U}$ but $\mathbf{U}_1 \cap \mathbf{Q}_2 = 1$. Let $\mathbf{U}_n$ be the root group corresponding to the half circle that contains Q_n but not Q_{n-1}, so that $\mathbf{U}_n \leq \mathbf{U}$ but $\mathbf{U}_n \cap \mathbf{Q}_{n-1} = 1$.

It follows that $\mathbf{U} - \mathbf{Q}_i$ has codimension in $\mathbf{U}$ at least 1 for $i = 2, n-1$. Since $\mathbf{U}(\mathcal{O}_K)$ is Zariski dense in $\mathbf{U}$, there is some $u \in \mathbf{U}(\mathcal{O}_K) - (\mathbf{Q}_2 \cup \mathbf{Q}_{n-1})$. It follows that u fixes the edges Q_n and Q_1, but no other edges in the circle corresponding to $\mathcal{A}$.

Since u is a bounded element of G, it fixes a point in X. Therefore, u fixes a geodesic ray in X that limits to an interior point of the edge corresponding to Q_1 in the spherical building. Any such geodesic ray must contain a point in $\mathcal{A}$, which is to say that u fixes a point in $\mathcal{A}$.

Define a height function $q : \mathcal{A} \to \mathbb{R}$ such that the pre-image of any point is an axis of translation for b, such that $s \leq t$ if and only if any geodesic ray in $\mathcal{A}$ that eminates from $q^{-1}(s)$ and limits to P contains a point from $q^{-1}(t)$.

Let $F = \{ x \in \mathcal{A} \mid ux = x \}$, let $I = \inf_{f \in F} \{ q(f) \}$, and let $E = \{ f \in F \mid q(f) = I \}$. Since the fixed set of u in the circle at infinity of $\mathcal{A}$ equals the union of the two edges Q_1 and Q_n, and since F is convex, I exists and E is either a point of, a subray of, a line segment of, or an entire axis of translation for b.

Notice that E is bounded, otherwise u would fix the point at infinity that a subray of E limited to. This point at infinity would have distance $\pi/2$ from the vertex P in the spherical metric, but this is not possible as the previously identified fixed set of u in the boundary circle is centered at P and has radius at most $\pi/3$. (The bound $\pi/3$ is realized exactly when the root system for G is of type A_2.) Thus E is either a point or a compact interval.

Since the fix set of u in the boundary circle is exactly the union of Q_1 and Q_n, and since F is convex, F is precisely the union of all geodesic rays eminating from points in E and limiting to points in the arc $Q_1 \cup Q_n$. That is F is a polyhedral region in $\mathcal{A}$ that is symmetric with respect to a reflection of $\mathcal{A}$ through a geodesic that limits to P

and the opposite point of P. If E is a point, then F has two geodesic rays as its boundary: one ray that limits to P_2, and the other that limits to P_n. If E is a nontrivial interval, then the boundary of F is the union of E, a ray from an endpoint of E that limits to P_2, and a ray from the other endpoint of E that limits to P_n.

If E is an interval, we label its endpoints e^+ and e^- such that E is both oriented in the direction of translation of b, and in the direction towards e^+, and away from e^-. Let e_0 be the midpoint of E. If E is a point, then $e_0 = e^+ = e^-$ is that point.

For n_0 sufficiently large and for any $n \geq n_0$, we define $\sigma_n \subseteq \mathcal{A}$ as the geodesic segment between $b^{-n}e^+$ and $b^n e^-$. Notice that $b^{-n}e^+$ is the only point in σ_n that is fixed by $g_n = b^{-n}ub^n$, and that $b^n e^-$ is the only point in σ_n that is fixed by $h_n = b^n ub^{-n}$.

Recall that $\mathcal{A}$ is the apartment corresponding to A and $\mathbf{T} \subset \mathbf{A}$ is a K-split one dimensional torus of $\mathbf{G}$. Recall also that $\mathbf{P} = \mathbf{U}Z_{\mathbf{G}}(\mathbf{T})$. Let $a \in T$ be such that $a^{-n}ua^n \to 1$ as $n \to \infty$ so that $a^n e_0$ converges to the cell at infinity corresponding to P as $n \to \infty$.

Let Δ_n be the triangle with one face equal to σ_n, a second face contained in the boundary of $b^{-n}\mathrm{Fix}_{\mathcal{A}}(u) = \mathrm{Fix}_{\mathcal{A}}(g_n)$, a third face contained in the boundary of $b^n\mathrm{Fix}_{\mathcal{A}}(u) = \mathrm{Fix}_{\mathcal{A}}(h_n)$, and vertices $b^n e^-$, $b^{-n}e^+$, and a uniquely determined point $y_n \in \partial\mathrm{Fix}_{\mathcal{A}}(g_n) \cap \partial\mathrm{Fix}_{\mathcal{A}}(h_n)$. Thus y_n converges to the cell at infinity corresponding to P as $n \to \infty$.

Note that

1. $\mathbf{U}$ is a unipotent group so $[[[[g_n, h_n], \cdots], h_n], h_n] = 1$ for some fixed number of nested commutators that's independent of n.
2. If w is a word in $\{g_n, h_n, g_n^{-1}, h_n^{-1}\}$ and $d \in \{g_n, h_n, g_n^{-1}, h_n^{-1}\}$, then $w\sigma_n$ and $wd\sigma_n$ are incident.

(1) and (2) imply that the word $[[[[g_n, h_n], \cdots], h_n], h_n]$ (or possibly a subword) describes a 1-cycle that is the union of translates of σ_n by subwords of $[[[[g_n, h_n], \cdots], h_n], h_n]$. We name this 1-cycle c_n.

The cone of c_n at the point y_n is the topological image of a 2-disk $\phi_n : D^2 \to X$ such that $\phi_n(\partial D^2) = c_n$.

If we let

$$X_0 = \Gamma\sigma_{n_0}$$

then clearly $c_n \in X_0$ for all n since $b, g_n, h_n \in \Gamma$ and $\sigma_n \subseteq \langle b\rangle\sigma_{n_0}$.

6. Proof of Theorem 1

We choose a Γ-invariant and cocompact space $X_i \subseteq X$ to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq \cup_{i=1}^{\infty} X_i = X$$

In our present context, Brown's criterion takes on the following form [Br]

Brown's Filtration Criterion 7. *By Lemma 2, the group Γ is not of type FP_2 (and hence not finitely presented) if for any $i \in \mathbb{N}$, there exists some class in the homology group $\widetilde{H}_1(X_0\,, \mathbb{Z})$ which is nonzero in $\widetilde{H}_1(X_i\,, \mathbb{Z})$.*

Since $\Gamma \backslash X_i$ is compact it follows from Lemma 3 that for any i there there exists some j_i such that $a^{j_i}e_0 \notin X_i$. Choose n sufficiently large so that $a^{j_i}e_0 \in \Delta_n \subseteq \phi_n$. Recall that $c_n \subseteq X_0$. Since X is contractible and 2-dimensional, any filling disk for c_n must contain $a^{j_i}e_0$. That is, c_n represents a nontrivial class in the homology of $X - \{a^{j_i}e_0\}$, and hence is nontrivial in the homology of X_i.

7. Other ranks

The proof of Proposition 4.1 in [B-W] gives a short proof that $\mathbf{SL_2}(\mathbb{Z}[t])$ is not finitely generated by examining the action of $\mathbf{SL_2}(\mathbb{Z}[t])$ on the tree for $\mathbf{SL_2}(\mathbb{Q}((t^{-1})))$. Replacing some of the remarks for $\mathbf{SL_2}(\mathbb{Z}[t])$ in that paper with straightforward analogues from lemmas in this paper, it is easy to see that the proof in [B-W] applies to show that if $\mathbf{H}$ is a connected, noncommutative, absolutely almost simple algebraic K-group of K-rank 1, then $\mathbf{H}(\mathcal{O}_K[t])$ is not finitely generated.

It seems natural to state the following

Conjecture 1. *Suppose $\mathbf{H}$ is a connected, noncommutative, absolutely almost simple algebraic K-group whose K-rank equals k. Then $\mathbf{H}(\mathcal{O}_K[t])$ is not of type F_k or FP_k.*

The conjecture has been verified when $K = \mathbb{Q}$ and $\mathbf{H} = \mathbf{SL_n}$ [B-M-W].

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